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Energy-Momentum Complex in Møller's Tetrad Theory of Gravitation

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Møller's tetrad theory of gravitation is examined with regard to the energy-momentum complex. The energy-momentum complex as well as the superpotential associated with Møller's theory are derived. Møller's field equations are solved in the case of spherical symmetry. Two *different* solutions, giving rise to the *same* metric, are obtained. The energy associated with one solution is found to be twice the energy associated with the other. Some suggestions to get out of this inconsistency are discussed at the end of the paper.

1. INTRODUCTION

The problem of defining an energy-momentum complex describing the energy contents of physical systems in general relativity (GR) has been tackled by several authors (Einstein, 1916; Bergmann and Thomson, 1953; Goldberg, 1958). Møller (1958, 1961a, b) pointed out that all expressions proposed previously for this quantity have some defects. He specified some properties that need to be satisfied. Møller (1961b) has shown that it is not possible to get an expression with these specifications using Riemannian space. Instead, he suggested using tetrad space. In fact, he was able to derive an expression for the energy-momentum complex, possessing the properties mentioned before, in tetrad space.

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The Lagrangian function from which the field equations of GR are derived is invariant under local tetrad rotation. Thus the field equations do not fix the field variables completely, leaving undefined six free functions. As a consequence, many different tetrad structures may give rise to the same metric specifying the gravitational field. And since the energy-momentum complex suggested by Møller is not invariant under local tetrad rotation, a certain metric which is supposed to represent a single definite physical system may be associated with more than one quantity expressing its energy and momentum contents. Thus the problem was not solved completely by the proposed expression mentioned above. Møller (1961b) suggested that the field equations of GR have to be modified in order not to allow such redundancy in solutions.

Møller (1978) modified GR by constructing a new field theory in the tetrad space. The field equations in this new theory were derived from a Lagrangian which is *not* invariant under local tetrad rotation. This theory has gained considerable attention (Sáez, 1983, 1984, 1985; Sáez and de Juan, 1984; Meyer, 1982). The purpose of the present work is to examine this theory with regard to the energy-momentum complex proposed by Møller (1961b). In Section 2 we will review briefly Møller's tetrad theory of gravitation. The energy-momentum complex associated with Møller's theory is derived in Section 3. The structure of tetrad spaces with spherical symmetry is reviewed in Section 4. Two solutions of Møller's field equations are obtained in Section 5, using the tetrad of Section 4. A comparison between the two solutions is given in Section 6. In Section 7 the energy contents associated with each solution are evaluated. The results are discussed and concluded in Section 8.

Due to the lengthy and tedious calculations of the present work, the computer algebra system REDUCE 3.3 was used. Copies of the programs used are available.[5]

2. MØLLER'S TETRAD THEORY OF GRAVITATION

Møller (1978) constructed a gravitational theory using the tetrad space for its structure. His aim was to get a theory free from singularities while retaining the principle merits of GR as far as possible. In his theory the field variables are the 16 tetrad components $\underset{i}{h}^{\mu}$.[6]

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[6]In the following we use Latin indices ($ijk\ldots$) to represent the vector numbers, and Greek indices ($\mu\nu\sigma\ldots$) to represent the vector components. All indices run from 0 to 3.

The metric is a derived quantity, given by

$$g^{\mu\nu} \stackrel{\text{def}}{=} \underset{i}{h}^{\mu} \underset{i}{h}^{\nu} \tag{2.1}$$

We assume imaginary values for the vector $\underset{0}{h}^{\mu}$ in order for the above metric to have a Lorentz signature. A central role in Møller's theory is played by the tensor

$$\gamma_{\mu\nu\sigma} \stackrel{\text{def}}{=} \underset{i}{h}_{\mu} \underset{i}{h}_{\nu;\sigma} \tag{2.2}$$

where the semicolon denotes covariant differentiation using the Christoffel symbols. Møller (1978) considered the Lagrangian L to be an invariant constructed from $\gamma_{\mu\nu\sigma}$ and $g_{\mu\nu}$. As he pointed out, the simplest possible independent expressions are

$$L^{(1)} \stackrel{\text{def}}{=} \Phi_{\mu}\Phi^{\mu}, \qquad L^{(2)} \stackrel{\text{def}}{=} \gamma_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}, \qquad L^{(3)} \stackrel{\text{def}}{=} \gamma_{\mu\nu\sigma}\gamma^{\sigma\nu\mu} \tag{2.3}$$

where Φ_{μ} is the basic vector defined by

$$\Phi_{\mu} \stackrel{\text{def}}{=} \gamma^{\nu}{}_{\mu\nu} \tag{2.4}$$

These expressions $L^{(i)}$ in (2.3) are homogeneous quadratic functions in the first-order derivatives of the tetrad field components.

Møller considered the simplest case, in which the Lagrangian L is a linear combination of the quantities $L^{(i)}$, i.e., the Lagrangian density is given by

$$\mathscr{L}_{\text{Møller}} \stackrel{\text{def}}{=} (-g)^{1/2}(\alpha_1 L^{(1)} + \alpha_2 L^{(2)} + \alpha_3 L^{(3)}) \tag{2.5}$$

where

$$g \stackrel{\text{def}}{=} \det(g_{\mu\nu}) \tag{2.6}$$

Here, Møller (1978) chooses the constants α_i such that his theory gives the same results as GR in the linear approximation of weak fields. According to his calculations, one can easily see that if we choose

$$\alpha_1 = -1, \qquad \alpha_2 = \lambda, \qquad \alpha_3 = 1 - 2\lambda \tag{2.7}$$

with λ equal to a free dimensionless parameter of order unity, the theory will be in agreement with GR to the first order of approximation. For

$\lambda=0$, Møller's theory is identical with Einstein's theory, but for $\lambda\neq 0$ the field equations take the following form[7]:

$$G_{\mu\nu}+H_{\mu\nu}=-\kappa T_{\mu\nu} \tag{2.8}$$

$$F_{\mu\nu}=0 \tag{2.9}$$

where

$$H_{\mu\nu} \stackrel{\text{def}}{=} \lambda[\gamma_{\alpha\beta\mu}\gamma^{\alpha\beta}{}_{\nu}+\gamma_{\alpha\beta\mu}\gamma_{\nu}{}^{\alpha\beta}+\gamma_{\alpha\beta\nu}\gamma_{\mu}{}^{\alpha\beta}+g_{\mu\nu}(\gamma_{\alpha\beta\sigma}\gamma^{\sigma\beta\alpha}-\tfrac{1}{2}\gamma_{\alpha\beta\sigma}\gamma^{\alpha\beta\sigma})] \tag{2.10}$$

and

$$F_{\mu\nu} \stackrel{\text{def}}{=} \lambda[\Phi_{\mu,\nu}-\Phi_{\nu,\mu}-\Phi_{\alpha}(\gamma^{\alpha}{}_{\mu\nu}-\gamma^{\alpha}{}_{\nu\mu})+\gamma_{\mu\nu}{}^{\alpha}{}_{;\alpha}] \tag{2.11}$$

Equations (2.9) are independent of the free parameter λ. On the other hand, the term $H_{\mu\nu}$ by which equations (2.8) deviate from Einstein's field equations increases with λ, which can be taken of order unity without destroying the first-order agreement with Einstein's theory in the case of weak fields.

3. ENERGY-MOMENTUM COMPLEX FOR MØLLER'S THEORY

Møller (1961b) was able to find a general expression for an energy-momentum complex $\mathcal{M}_{\mu}{}^{\nu}$ that possesses all the required satisfactory properties, and formed its superpotential $\mathcal{U}_{\mu}{}^{\nu\alpha}$ using the method of infinitesimal transformations,

$$\mathcal{M}_{\mu}{}^{\nu} \stackrel{\text{def}}{=} (-g)^{1/2}\,(T_{\mu}{}^{\nu}+t_{\mu}{}^{\nu})=\mathcal{U}_{\mu}{}^{\nu\alpha}{}_{,\alpha} \tag{3.1}$$

where

$$(-g)^{1/2}\,t_{\mu}{}^{\nu} \stackrel{\text{def}}{=} \frac{1}{2\kappa}\left[\frac{\partial\mathcal{L}}{\partial \underset{i}{h}{}^{\alpha}{}_{,\nu}}\underset{i}{h}{}^{\alpha}{}_{,\mu}-\delta_{\mu}{}^{\nu}\mathcal{L}\right] \tag{3.2}$$

and

$$\mathcal{U}_{\mu}{}^{\nu\alpha} \stackrel{\text{def}}{=} \frac{1}{4\kappa}\left[\frac{\partial\mathcal{L}}{\partial \underset{i}{h}{}^{\mu}{}_{,\alpha}}\underset{i}{h}{}^{\nu}-\frac{\partial\mathcal{L}}{\partial \underset{i}{h}{}^{\mu}{}_{,\nu}}\underset{i}{h}{}^{\alpha}\right] \tag{3.3}$$

where $\mathcal{L}$ is the Lagrangian of the theory under consideration.

[7]Obviously $G_{\mu\nu}$ denotes the Einstein tensor, $T_{\mu\nu}$ denotes the material-energy tensor, and κ is the constant of GR, which equals 8π in relativistic units.

For Møller's Lagrangian, as given by (2.5), the superpotential (3.3) can be written in the form

$$\mathcal{U}_\mu{}^{\nu\sigma} = \frac{(-g)^{1/2}}{4\kappa} [\alpha_1 \overset{(1)}{U}_\mu{}^{\nu\sigma} + \alpha_2 \overset{(2)}{U}_\mu{}^{\nu\sigma} + \alpha_3 \overset{(3)}{U}_\mu{}^{\nu\sigma}] \tag{3.4}$$

where $\overset{(1)}{U}_\mu{}^{\nu\sigma}$, $\overset{(2)}{U}_\mu{}^{\nu\sigma}$, $\overset{(3)}{U}_\mu{}^{\nu\sigma}$ correspond to $L^{(1)}$, $L^{(2)}$, $L^{(3)}$, respectively.

To evaluate the superpotential we have first (see Appendix A in Møller, 1961b)

$$\partial \underset{i}{h}_{\mu;\nu} / \partial \underset{j}{h}^\rho{}_{,\tau} = -\tfrac{1}{2} \underset{i}{h}^\alpha P_{\alpha\mu\nu\rho}{}^{\sigma\tau} \underset{j}{h}_\sigma \tag{3.5}$$

where $P_{\alpha\mu\nu}{}^{\rho\sigma\tau}$ is a tensor of the form

$$P_{\alpha\mu\nu}{}^{\rho\sigma\tau} \overset{\text{def}}{=} \delta_\alpha{}^\rho g_{\mu\nu}{}^{\sigma\tau} + \delta_\mu{}^\rho g_{\nu\alpha}{}^{\sigma\tau} - \delta_\nu{}^\sigma g_{\alpha\mu}{}^{\sigma\tau} \tag{3.6}$$

and $g_{\mu\nu}{}^{\sigma\tau}$ is the tensor

$$g_{\mu\nu}{}^{\sigma\tau} \overset{\text{def}}{=} \delta_\mu{}^\sigma \delta_\nu{}^\tau - \delta_\nu{}^\sigma \delta_\mu{}^\tau \tag{3.7}$$

Thus we get

$$\begin{aligned}
\frac{\partial L^{(1)}}{\partial \underset{i}{h}^\mu{}_{,\sigma}} &= g^{\alpha\beta} \frac{\partial}{\partial \underset{i}{h}^\mu{}_{,\sigma}} \Phi_\alpha \Phi_\beta \\
&= 2\Phi^\alpha \frac{\partial}{\partial \underset{i}{h}^\mu{}_{,\sigma}} \Phi_\alpha \\
&= 2\Phi^\alpha \underset{j}{h}^\beta \frac{\partial}{\partial \underset{i}{h}^\mu{}_{,\sigma}} \underset{ij}{h}_{\alpha;\beta} \\
&= 2\Phi^\alpha \underset{j}{h}^\beta \left[-\frac{1}{2} \underset{ij}{h}^\varepsilon P_{\varepsilon\alpha\beta\mu}{}^{\rho\sigma} \underset{i}{h}_\rho \right] \\
&= -\Phi^\alpha g^{\beta\varepsilon} P_{\varepsilon\alpha\beta\mu}{}^{\rho\sigma} \underset{i}{h}_\rho
\end{aligned} \tag{3.8}$$

Finally, we get

$$\begin{aligned}
\overset{(1)}{U}_\mu{}^{\nu\sigma} &\overset{\text{def}}{=} \frac{\partial L^{(1)}}{\partial \underset{i}{h}^\mu{}_{,\sigma}} \underset{i}{h}^\nu - \frac{\partial L^{(1)}}{\partial \underset{i}{h}^\mu{}_{,\nu}} \underset{i}{h}^\sigma \\
&= -2\Phi^\alpha g^{\beta\varepsilon} g_{\mu\tau} P_{\varepsilon\alpha\beta}{}^{\tau\nu\sigma}
\end{aligned} \tag{3.9}$$

Similarly we can write

$$\overset{(2)}{U}_\mu{}^{\nu\sigma} = -2\gamma^{\varepsilon\alpha\beta} g_{\mu\tau} P_{\varepsilon\alpha\beta}{}^{\tau\nu\sigma} \tag{3.10}$$

$$\overset{(3)}{U}_\mu{}^{\nu\sigma} = -2\gamma^{\beta\alpha\varepsilon} g_{\mu\tau} P_{\varepsilon\alpha\beta}{}^{\tau\nu\sigma} \tag{3.11}$$

The final expression for the superpotential for Møller's theory can be obtained by substituting from (3.9)–(3.11) and using the values of the parameters $\alpha_1, \alpha_2, \alpha_3$ given in Section 2 in (3.4), to get

$$\mathcal{U}_\mu{}^{\nu\sigma} = \frac{(-g)^{1/2}}{2\kappa} P_{\varepsilon\alpha\beta}{}^{\tau\nu\sigma}[\Phi^\alpha g^{\beta\varepsilon} g_{\mu\tau} - \lambda g_{\tau\mu}\gamma^{\varepsilon\alpha\beta} - (1-2\lambda)\, g_{\tau\mu}\gamma^{\beta\alpha\varepsilon}] \tag{3.12}$$

4. SPHERICALLY SYMMETRIC TETRAD SPACES

The structure of tetrad spaces with spherical symmetry has been studied by Robertson (1932). The four tetrad vectors defining such structure, as given by Robertson, can be written as

$$\begin{aligned} \underset{0}{h}{}^0 &= A \\ \underset{0}{h}{}^a &= DX^a \\ \underset{a}{h}{}^0 &= EX^a \\ \underset{a}{h}{}^b &= FX^aX^b + \delta_a{}^b B + \varepsilon_{abc} SX^c \end{aligned} \tag{4.1}$$

where A, B, D, E, F, and S are functions of $r = (X^aX^a)^{1/2}$ and a, b, c run from 1 to 3.

Robertson has shown that:

1. Improper rotations are admitted if and only if $S = 0$. In this case the tetrad (4.1) takes the form

$$\begin{aligned} \underset{0}{h}{}^0 &= A \\ \underset{0}{h}{}^a &= DX^a \\ \underset{a}{h}{}^0 &= EX^a \\ \underset{a}{h}{}^b &= FX^aX^b + \delta_a{}^b B \end{aligned} \tag{4.2}$$

2. The functions E and F can be eliminated by mere coordinate transformations, leaving the tetrad in the simpler form

$$\begin{aligned} \underset{0}{h}{}^0 &= A \\ \underset{0}{h}{}^a &= DX^a \\ \underset{a}{h}{}^b &= \delta_a{}^b B \end{aligned} \tag{4.3}$$

Three important remarks are reported here:

1. The tetrad used by Møller (1978) in application of his theory is a special case of the above tetrad (4.3), in which the function D is taken to be zero. Thus one may expect to obtain more solutions when using the more general tetrad (4.3).

2. Since one has to take the vector $\underset{0}{h}^{\mu}$ to be imaginary, in order to preserve the Lorentz signature for the metric, the functions A and D have to be taken as imaginary.

3. It is more convenient, for the sake of computations, to use the tetrad (4.3) in spherical polar coordinates, where it takes the form[8]

$$\underset{i}{h}^{\mu}=\begin{pmatrix} A & Dr & 0 & 0 \\ 0 & B\sin\theta\cos\phi & \frac{B}{r}\cos\theta\cos\phi & -\frac{B\sin\phi}{r\sin\theta} \\ 0 & B\sin\theta\sin\phi & \frac{B}{r}\cos\theta\sin\phi & \frac{B\cos\phi}{r\sin\theta} \\ 0 & B\cos\theta & -\frac{B}{r}\sin\theta & 0 \end{pmatrix} \tag{4.4}$$

5. SOLUTIONS OF MØLLER'S FIELD EQUATIONS

Using the tetrad (4.4) to solve Møller's field equations (2.8) and (2.9), we find that equation (2.9) is satisfied identically, and also that $H_{\mu\nu}$ as given by (2.10) vanishes identically. Thus for spherically symmetric exterior solutions, Møller's field equations are reduced to Einstein's field equations of GR, namely

$$G_{\mu\nu}=0 \tag{5.1}$$

The Einstein tensor $G_{\mu\nu}$ may be evaluated using the Riemannian metric derived from (4.4) via the relation (2.1). It is easy to get

$$\begin{aligned} &g_{00}=\frac{B^2+D^2r^2}{A^2B^2}, \qquad g_{10}=g_{01}=-\frac{Dr}{AB^2}, \qquad g_{11}=\frac{1}{B^2} \\ &g_{22}=\frac{r^2}{B^2}, \qquad g_{33}=\frac{r^2\sin^2\theta}{B^2} \end{aligned} \tag{5.2}$$

[8]In matrix notation, the element $\underset{i}{h}^{\mu}$ is in the ith row and the μth column.

The corresponding field equations (5.1) give rise to the following set of differential equations:

$$G_{00} = -\frac{1}{rA^2B^4}\{[(3D^2+8B'^2)D-2(2DB''+B'D')B]r^3B^2D$$
$$-[2(DB''+B'D')B-5DB'^2]r^5D^3-(2BB''-3D^2-3B'^2)rB^4$$
$$+2(BD'-4DB')r^4BD^3+2(BD'-6DB')r^2B^3D-4B^5B'\}=0 \qquad (5.3)$$

$$G_{01} = -\frac{D}{AB^4}\{[2(DB''+B'D')B-5DB'^2]r^3D+(2BB''-3D^2-3B'^2)rB^2$$
$$-2(BD'-4DB')r^2BD+4B^3B'\}=0 \qquad (5.4)$$

$$G_{11} = -\frac{1}{rAB^4}\{[(3D^2+B'^2)A+2BA'B']rB^2$$
$$-[2(DB''+B'D')B-5DB'^2]r^3AD$$
$$+2(BD'-4DB')r^2ABD-2AB^3B'-2B^4A'\}=0 \qquad (5.5)$$

$$G_{22} = -\frac{r}{A^2B^4}[\{[(DA''+3A'D')B-3DA'B']ABD$$
$$+[(2DB''+5B'D')BD-(DD''+D'^2)B^2-5D^2B'^2]A^2-2B^2D^2A'^2\}r^3$$
$$+[(BB''-3D^2B'^2)A^2+AB^2A''-2B''A'^2]rB^2$$
$$-2[(3BD'-4DB')A-2BDA']r^2ABD+A^2B^3B'+AB^4A')]=0 \quad (5.6)$$

$$G_{33} = -\frac{r\sin^2\theta}{A^2B^4}\times[(\{[(DA''+3A'D')B-3DA'B']ABD$$
$$+[(2DB''+5B'D')BD-(DD''+D'^2)B^2-5D^2B'^2]A^2-2B^2D^2A'^2\}r^3$$
$$+[(BB''-3D^2B'^2)A^2+AB^2A''-2B''A'^2)rB^2$$
$$-2[(3BD'-4DB')A-2BDA']r^2ABD+A^2B^3B'+AB^4A')]=0 \quad (5.7)$$

where the primes refers to differentiation with respect to r.

The trivial flat space-time solution for such equations is obtained by taking

$$A=i, \qquad B=1, \qquad D=0 \qquad (5.8)$$

A first nontrivial solution can be obtained by taking $D=0$ and solving for A and B. In fact, this is the case studied by Møller (1978), where he obtained the solution

$$A=i\frac{1+m/2r}{1-m/2r}, \qquad B=\frac{1}{(1+m/2r)^2} \qquad (5.9)$$

Hence, we get from (4.3) directly the tetrad (in Cartesian coordinates)

$$\underset{0}{h}^{0} = i\,\frac{1+m/2r}{1-m/2r}$$
$$\underset{1}{h}^{1} = \underset{2}{h}^{2} = \underset{3}{h}^{3} = \frac{1}{(1+m/2r)^2} \tag{5.10}$$

with the associated Riemannian metric

$$ds^2 = -\frac{(1-m/2r)^2}{(1+m/2r)^2}\,dt^2 + \left(1+\frac{m}{2r}\right)^4 (dX^2 + dY^2 + dZ^2) \tag{5.11}$$

i.e., the Schwarzschild metric in its isotropic form.

A second nontrivial solution can be obtained by taking $A = i$, $B = 1$, and $D \neq 0$ and solving for D. In this case the resulting field equations can be integrated directly to give

$$D = i\left(\frac{2m}{r^3}\right)^{1/2} \tag{5.12}$$

Hence, we get from (4.3) the following tetrad (in Cartesian coordinates):

$$\underset{0}{h}^{0} = i$$
$$\underset{0}{h}^{a} = i\left(\frac{2m}{r^3}\right)^{1/2} X^a \tag{5.13}$$
$$\underset{a}{h}^{b} = \delta_a{}^b$$

where a and b run from 1 to 3. The metric associated with the above tetrad is

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 - 2\left(\frac{2m}{r^3}\right)^{1/2} X\,dt\,dX - 2\left(\frac{2m}{r^3}\right)^{1/2} Y\,dt\,dY$$
$$-2\left(\frac{2m}{r^3}\right)^{1/2} Z\,dt\,dZ + dX^2 + dY^2 + dZ^2 \tag{5.14}$$

A simpler form for the above metric can be obtained if it is written in polar coordinates. Substituting directly in (4.4) for the value of D as given by (5.12), we get the tetrad (in polar coordinates)

$$h_i^{\mu} = \begin{pmatrix} i & i\left(\frac{2m}{r}\right)^{1/2} & 0 & 0 \\ 0 & \cos\phi\sin\theta & \frac{\cos\phi\cos\theta}{r} & -\frac{\sin\phi}{r\sin\theta} \\ 0 & \sin\phi\sin\theta & \frac{\cos\theta\sin\phi}{r} & \frac{\cos\phi}{r\sin\theta} \\ 0 & \cos\theta & -\frac{\sin\theta}{r} & 0 \end{pmatrix} \quad (5.15)$$

with the associated Riemannian metric

$$ds^2 = -(1-2m/r)\,dt^2 - 2(2m/r)^{1/2}\,dt\,dr + dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \quad (5.16)$$

It is to be noted here that m in the metric (5.16) is a mere constant of integration. It will be shown in the next section that m indeed plays the role of the mass producing the field, and thus justifies the use of its name.

6. COMPARISON OF THE TWO SOLUTIONS

Our aim in this section is to compare the two different solutions obtained in Section 5 for Møller's field equations. The first step is to eliminate the cross term appearing in the metric (5.16) of the second solution. This can be easily done by performing the coordinate transformation

$$t \to t + \int \frac{-iDr}{(1-D^2r^2)}\,dr \quad (6.1)$$

and keeping the spatial coordinates unchanged. One gets the transformed tetrad in the form

$$\begin{aligned} &\underset{0}{h}^0 = \frac{i}{1-2m/r} && \underset{0}{h}^1 = i\left(\frac{2m}{r}\right)^{1/2} \\ &\underset{1}{h}^0 = \left(\frac{2m}{r}\right)^{1/2}\frac{\cos\phi\sin\theta}{1-2m/r}, && \underset{1}{h}^1 = \cos\phi\sin\theta \\ &\underset{1}{h}^2 = \frac{\cos\phi\cos\theta}{r}, && \underset{1}{h}^3 = -\frac{\sin\phi}{r\sin\theta} \\ &\underset{2}{h}^0 = \left(\frac{2m}{r}\right)^{1/2}\frac{\sin\phi\sin\theta}{1-2m/r}, && \underset{2}{h}^1 = \sin\phi\sin\theta \\ &\underset{2}{h}^2 = \frac{\cos\theta\sin\phi}{r}, && \underset{2}{h}^3 = \left(\frac{2m}{r}\right)^{1/2}\frac{\cos\theta}{1-2m/r} \\ &\underset{3}{h}^1 = \cos\theta, && \underset{3}{h}^2 = -\frac{\sin\theta}{r} \end{aligned} \quad (6.2)$$

The metric associated with the above tetrad can be computed either directly from the tetrad or by applying the same coordinate transformation (6.1) to the metric (5.16). In both cases we get

$$ds^2 = -(1-2m/r)\,dt^2 + (1-2m/r)^{-1}\,dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \quad (6.3)$$

i.e., the Schwarzschild metric in its standard form, in which m, the constant of integration, plays the role of the mass of the source of the field.

Now to be able to compare the two solutions (5.10) and (6.2), we transform the second one (6.2) to a coordinate system such that its Riemannian metric takes the isotropic form in Cartesian coordinates, i.e., the same coordinates of the first solution (6.2). The first coordinate transformation (Eddington, 1921, p. 93) is

$$r \to r\left(1+\frac{m}{2r}\right)^2 \quad (6.4)$$

Applying this coordinate transformation yields the following tetrad:

$$\underset{0}{h}{}^0 = i\,\frac{(1+m/2r)^2}{(1-m/2r)^2},$$

$$\underset{0}{h}{}^1 = i\,\frac{2(m/2r)^{1/2}}{(1-m/2r)(1+m/2r)^2}$$

$$\underset{1}{h}{}^0 = 2\left(\frac{m}{2r}\right)^{1/2} \frac{(1+m/2r)\cos\phi\sin\theta}{(1-m/2r)^2},$$

$$\underset{1}{h}{}^1 = \frac{\cos\phi\sin\theta}{(1-m/2r)(1+m/2r)}$$

$$\underset{1}{h}{}^2 = \frac{\cos\phi\cos\theta}{r(1+m/2r)^2},$$

$$\underset{1}{h}{}^3 = -\frac{\sin\phi}{r\sin\theta(1-m/2r)^2}$$

$$\underset{2}{h}{}^0 = 2\left(\frac{m}{2r}\right)^{1/2} \frac{(1+m/2r)\sin\phi\sin\theta}{(1-m/2r)^2},$$

$$\underset{2}{h}{}^1 = \frac{\sin\phi\sin\theta}{(1+m/2r)(1-m/2r)}$$

$$\underset{2}{h}{}^2 = \frac{\cos\theta\sin\phi}{r(1+m/2r)^2},$$

$$\underset{2}{h}^3 = \frac{\cos\phi}{r\sin\theta(1+m/2r)^2}$$

$$\underset{3}{h}^0 = 2\left(\frac{m}{2r}\right)^{1/2}\frac{(1+m/2r)\cos\theta}{(1-m/2r)^2},$$

$$\underset{3}{h}^1 = \frac{\cos\theta}{(1+m/2r)(1-m/2r)}$$

$$\underset{3}{h}^3 = -\frac{\sin\theta}{r(1+m/2r)^2} \tag{6.5}$$

The metric associated with the tetrad (6.5) is

$$ds^2 = -\frac{(1-m/2r)^2}{(1+m/2r)^2}\,dt^2 + \left(1+\frac{m}{2r}\right)^4 (dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2) \tag{6.6}$$

The last step is to transform the tetrad (6.5), along with its metric (6.6), into Cartesian coordinates,

$$\underset{0}{h}^0 = i\frac{(1+m/2r)^2}{(1-m/2r)^2},$$

$$\underset{0}{h}^1 = i\frac{2(m/2r)^{1/2}\,X}{r(1-m/2r)(1+m/2r)^2}$$

$$\underset{0}{h}^2 = i\frac{2(m/2r)^{1/2}\,Y}{r(1-m/2r)(1+m/2r)^2},$$

$$\underset{0}{h}^3 = i\frac{2(m/2r)^{1/2}\,Z}{r(1-m/2r)(1+m/2r)^2}$$

$$\underset{1}{h}^0 = \frac{2(2m/r)^{1/2}\,(1+m/2r)\,X}{(1-m/2r)^2\,r},$$

$$\underset{1}{h}^1 = \frac{(1-m/2r)\,r^2 + 2X^2(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2}$$

$$\underset{1}{h}^2 = \frac{2XY(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2},$$

$$\underset{1}{h}^3 = \frac{2XZ(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2}$$

$$\underset{2}{h}^0 = \frac{2(2m/r)^{1/2}\,(1+m/2r)\,Y}{(1-m/2r)^2\,r},$$

$$\underset{2}{h}^{1} = \frac{2XY(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2}$$

$$\underset{2}{h}^{2} = \frac{(1-m/2r)\,r^2 + 2Y^2(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2},$$

$$\underset{2}{h}^{3} = \frac{2YZ(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2}$$

$$\underset{3}{h}^{0} = \frac{2(2m/r)^{1/2}\,(1+m/2r)\,Z}{(1-m/2r)^2\,r},$$

$$\underset{3}{h}^{1} = \frac{2XZ(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2}$$

$$\underset{3}{h}^{2} = \frac{2YZ(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2},$$

$$\underset{3}{h}^{3} = \frac{(1-m/2r)\,r^2 + 2Z^2(m/2r)}{(1+m/2r)^2\,(1-m/2r)\,r^2} \tag{6.7}$$

The metric derived from the above tetrad (6.7) is now identical with the metric derived from the first solution (5.11), namely

$$ds^2 = -\frac{(1-m/2r)^2}{(1+m/2r)^2}\,dt^2 + \left(1+\frac{m}{2r}\right)^4 (dX^2 + dY^2 + dZ^2) \tag{6.8}$$

Thus we have two exact solutions of Møller field equations, each of which leads to the same metric, the Schwarzschild metric in its isotropic form in Cartesian coordinates. We notice that the second solution (6.7) is of the form of the original Robertson tetrad (4.2). This should be expected, since the coordinate transformations we have performed on the second solution reproduce the functions E and F, eliminated before by coordinate transformation. Hence, we can put these two solutions into a concise form, as shown in Table I.

The important result obtained in this section is that we have been able to derive two different solutions for Møller's field equations; the Riemannian metrics associated with these two solutions are identical, namely the Schwarzschild metric in its isotropic form. Since Møller's theory is a pure gravitational theory, the above two solutions have to be equivalent in the sense that they describe the same physical situation, a static, spherically symmetric gravitational field with a source of mass m. In what follows we examine the equivalence of these solutions by calculating the energy associated with each of them, using the superpotential derived for Møller's theory in Section 3.

Table I. Comparison of the Two Solutions

Function	First solution (5.10)	Second solution (6.7)
A	$i\dfrac{1+m/2r}{1-m/2r}$	$i\dfrac{(1+m/2r)^2}{(1-m/2r)^2}$
B	$\dfrac{1}{(1+m/2r)^2}$	$\dfrac{1}{(1+m/2r)^2}$
D	0	$\dfrac{2i}{r}\left(\dfrac{m}{2r}\right)^{1/2}\dfrac{1}{(1+m/2r)^2\,(1-m/2r)^2}$
E	0	$\dfrac{2}{r}\left(\dfrac{m}{2r}\right)^{1/2}\dfrac{1+m/2r}{(1-m/2r)^2}$
F	0	$\dfrac{2}{r}\dfrac{m/2r}{(1+m/2r)^2\,(1-m/2r)}$

7. THE ENERGY ASSOCIATED WITH EACH SOLUTION

Now we use the superpotential of Møller's theory derived in Section 3 to evaluate the energy associated with each of the two solutions given in Table I. The components of the superpotential that contribute to the total energy are $\mathcal{U}_0{}^{0\sigma}$ only. Thus, substituting from the first solution (5.10) into (3.12), we get the following nonvanishing value

$$\mathcal{U}_0{}^{0a} = \frac{4X^a}{\kappa r^2}\frac{m}{2r}\left(1-\frac{m}{2r}\right) \tag{7.1}$$

The total energy is given by (Møller, 1958)

$$E = \lim_{r\to\infty}\int_{r=\mathrm{const}} \mathcal{U}_0{}^{0a} n_a\, dS \tag{7.2}$$

where n_a is a unit 3-vector normal to the surface elements dS. Substituting from (7.1) into (7.2), we get

$$E = \frac{8\pi m}{\kappa} = m \tag{7.3}$$

In nonrelativistic units, the above result appears as the mass of the source times the square of the speed of light. This is a very satisfactory result, and it should be expected.

Now let us turn our attention to the second tetrad (6.7). Computing the required components of the superpotential, we get

$$\mathcal{U}_0^{\ 0a} = \frac{8X^a}{\kappa r^2} \frac{m}{2r} \tag{7.4}$$

These lead to a total energy

$$E = 2m \tag{7.5}$$

This is twice the gravitational mass!

8. DISCUSSION AND CONCLUSION

The energy-momentum complex for Møller's tetrad theory of gravitation is derived, using Møller's Lagrangian. Two different exact solutions of Møller's field equations are obtained for the case of spherical symmetry. The energy content of each solution is evaluated using the derived superpotential. It is shown that, although the two solutions give rise to the same Riemannian metric (the Schwarzschild metric), they give two different values for the energy content. This shows a certain inconsistency in Møller's theory.

The following suggestions may be considered to get out of this inconsistency:

1. The energy-momentum complex suggested by Møller (1961b) is not quite adequate, though it has very satisfying properties.

2. Many authors believe that a tetrad theory should describe more than a pure gravitational field. In fact, Møller (1961b) considered this possibility in his earlier trials to modify GR. In these theories, the most successful candidates for the description of the other physical phenomenon are the skew-symmetric tensors of the tetrad space, e.g., $\Phi_{\mu;\nu} - \Phi_{\nu;\mu}$. *The most striking remark here is that all the skew-symmetric tensors vanish for the first solution, but not all of them do so for the second one.* Some authors (e.g., Einstein, 1930; Mikhail and Wanas, 1977) believe that these tensors are related to the presence of an electromagnetic field. Others (e.g., Müller-Hoissen and Nitsch, 1983) believe that these tensors are closely connected to the spin phenomenon. It is not clear that Møller's theory deserves such a wider interpretation. This needs a lot of investigation before arriving at a concrete conclusion.

3. The last possibility is that Møller's theory needs to be generalized rather than reinterpreted. There are already some generalizations of Møller's theory. Møller himself considered this possibility at the end of his 1978 paper by including terms in the Lagrangian other than the simple

quadratic terms $L^{(i)}$. Sáez (1983) has generalized Møller's theory in a very elegant and natural way into scalar tetradic theories of gravitation. In these theories the question is: Do the field equations fix the tetradic geometry in the case of spherical symmetry? This question was discussed at length by Sáez (1986). The results of the present paper can be considered as a first step to get a satisfactory answer to this question. Meyer (1982) showed that Møller's theory is a special case of Poincaré gauge theory constructed by Hehl *et al.* (1980). Thus Poincaré gauge theory can be considered as another satisfactory generalization of Møller's theory.

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